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Non-Abelian pp-waves in $D = 4$ supergravity theories

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ABSTRACT

The non-Abelian plane waves, first found in flat spacetime by Coleman and subsequently generalized to give pp-waves in Einstein-Yang-Mills theory, are shown to be $\frac{1}{2}$- supersymmetric solutions of a wide variety of $N = 1$ supergravity theories coupled to scalar and vector multiplets, including the theory of $SU(2)$ Yang-Mills coupled to an axion σ and dilaton ϕ recently obtained as the reduction to four-dimensions of the six-dimensional Salam-Sezgin model. In this latter case they provide the most general supersymmetric solution. Passing to the Riemannian formulation of this theory we show that the most general supersymmetric solution may be constructed starting from a self-dual Yang-Mills connection on a self-dual metric and solving a Poisson equation for e^{ϕ}. We also present the generalization of these solutions to non-Abelian AdS pp-waves which allow a negative cosmological constant and preserve $\frac{1}{4}$ of supersymmetry.

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1 Introduction

There is a considerable literature on Yang-Mills theory coupled to Einstein gravity in four-dimensions, but comparatively few exact results are known outside the realm of Abelian embeddings, where the problem reduces to the even more intensively studied Einstein-Maxwell theory. In the four-dimensional flat Minkowski spacetime, Coleman [1] exhibited genuinely non-abelian solutions of plane wave type and these were subsequently generalized to incorporate the effects of gravity [2], [3]. Non-abelian plane waves were then embedded into ten-dimensional Superstring Theory in [4] but the Yang-Mills field considered there did not arise through a process of dimensional reduction. It was found that these solutions constitute a natural generalization of the familiar vacuum pp-wave spacetimes. In the particular case of the non-Abelian plane waves it was also noted that, as in the case of the four-dimensional vacuum plane-waves [5] [6], the solutions are not affected by quantum corrections. In the Riemannian regime a great deal is known about self-dual Yang-Mills solutions on a fixed background, but these are not usually consistent unless the metric is Ricci flat. One particularly interesting case is the extension of the well-known 't Hooft ansatz to a Hyper-Kähler background apparently first written down in [7] (see [8] [9] for recent discussions). There is also a considerable literature on Einstein-Yang-Mills theory coupled to a dilaton, but again few exact results are known.

For physical interpretations it is natural to view these problems from the point of view of supersymmetry and of higher dimensions. The smallest dimension which yields a non-Abelian Yang-Mills theory through the Kaluza-Klein reduction is six and recently, two of us have shown [10] how the $SU(2)$ Yang-Mills fields together with an axion and a dilaton, coupled to the four-dimensional Einstein gravity can be obtained as a consistent and supersymmetric Pauli type reduction of the six-dimensional $(1, 0)$ gauged supergravity theory. This theory was first constructed by Nishino and Sezgin [12] and its reduction to four dimensions was first discussed by Salam and Sezgin [13]. The resulting four-dimensional theory contains a scalar multiplet whose bosonic sector comprises an axion and a dilaton together with an $SU(2)$ Yang-Mills vector multiplet. There is no potential for the scalars. The theory can be seen as a special case of the general coupling of scalar and vector multiplets to $N = 1$ supergravity first worked out by Cremmer et al. [14–16]. Under some circumstances, for example for purely magnetic or purely electric solutions, the axion field σ can be consistently set to a constant value in which case the equations of motion coincide with those of the much studied Einstein-Maxwell-dilaton system modulo the non-Abelian nature of the spin-1 fields. An interesting class of non-trivial Abelian solutions are of Robinson-Trautman

type [17].

In the study of all supersymmetric backgrounds of the six-dimensional gauged supergravity theory a sub-problem is to find all supersymmetric solutions of its four-dimensional Pauli reduction. One of the purposes of the present letter is to give the complete solution of this sub-problem and this will lead to an interesting connection to the old works on non-Abelian pp-waves. We shall show that the non-Abelian pp-waves exhaust the class of all supersymmetric solutions of the four-dimensional equations of the Salam-Sezgin model. From this result it follows that Coleman's original solutions exhaust the class of supersymmetric Yang-Mills fields in flat spacetime thus giving them a possible significance. Moreover, we shall show that non-Abelian pp-waves are in fact supersymmetric solutions of the much wider class of theories constructed by Cremmer et al. We shall also furnish the generalization of the non-Abelian pp-wave solutions which incorporates a negative cosmological constant. These will be the non-Abelian anti-de Sitter (AdS) pp-waves which may have potential applications to the AdS/CFT correspondence. We shall begin by considering the simplest Salam-Sezgin model.

2 The Salam-Sezgin Model

In a previous letter, three of us showed [18] that the remarkable supersymmetric background found by Salam and Sezgin is in fact unique among all non-singular solutions of the six-dimensional theory with four-dimensional Poincaré, de Sitter or anti-de Sitter invariance. The present section is to some extent complementary: we obtain all supersymmetric solutions of the four-dimensional reduced theory. The reduced four-dimensional theory describes $N = 1$ supergravity coupled to a scalar multiplet whose bosonic sector comprises an axion σ and dilaton ϕ and an $SU(2)$ vector multiplet whose bosonic sector consists of a Yang-Mills connection A^i.

The bosonic Lagrangian of this reduced theory is

$$\mathcal{L} = R \,{*\mathbb{1}} - \tfrac{1}{2}{*d\phi} \wedge d\phi - \tfrac{1}{2}e^{2\phi}\, {*d\sigma} \wedge d\sigma - \tfrac{1}{2}e^{-\phi}\, {*F^i} \wedge F^i + \tfrac{1}{2}\sigma\, F^i \wedge F^i \,, \tag{2.1}$$

and the four-dimensional fermionic supersymmetry transformations are

$$\delta\lambda^i \;=\; -\tfrac{1}{4\sqrt{2}}\, e^{-\frac{1}{2}\phi}\, F^i_{\alpha\beta}\, \gamma^{\alpha\beta}\epsilon \,, \tag{2.2}$$

$$\delta\chi \;=\; \tfrac{1}{4}(\partial_\alpha\phi\, - \mathrm{i}\, e^{\phi}\, \partial_\alpha\sigma\, \gamma_5)\gamma^\alpha\, \epsilon \,, \tag{2.3}$$

$$\delta\psi'_\alpha \;=\; \nabla_\alpha\, \epsilon - \tfrac{\mathrm{i}}{4}\, e^{\phi}\, \partial_\alpha\sigma\, \gamma_5\epsilon \,. \tag{2.4}$$

Since we are using spacetime signature $(-,+,+,+)$, there exists a representation of the gamma matrices such that they are all real. Our convention for γ_5 is such that $\gamma_5^2 = +1$, and hence $\mathrm{i}\,\gamma_5$ is also real in this representation. Thus we may regard the spinors of the four-dimensional theory either as purely real (i.e. Majorana) or as complex and Weyl.

Note that if $F^i \wedge F^i = 0$, then it is consistent to set the field σ to a constant value, and the equations of motion reduce to those of the pure Einstein-Yang-Mills-Dilaton system.

The consistent embedding of the above four-dimensional $N = 1$ theory into the six-dimensional chiral supergravity of Salam and Sezgin was obtained in [10]; in the bosonic sector, the reduction ansatz is given by

$$\begin{aligned}
d\hat{s}_6^2 &= e^{\frac{1}{2}\phi}\, ds_4^2 + e^{-\frac{1}{2}\phi}\, g_{mn}\, (dy^m + 2g\, K_i^m\, A^i)(dy^n + 2g\, K_j^n\, A^j)\,, \\
\hat{F}_{(2)} &= \frac{1}{2g}\, \Omega_2 - d(\mu^i\, A^i)\,, \\
\hat{H}_{(3)} &= H_{(3)} - 2g\, F^i \wedge K_{i\,m}\, (dy^m + 2g\, A^j\, K_j^m)\,, \\
\hat{\phi} &= -\phi\,,
\end{aligned} \qquad (2.5)$$

where $K_i^m = (8g^2)^{-1}\, \epsilon^{mn}\, \partial_n\, \mu^i$ are the Killing vectors on the unit 2-sphere defined by $\mu^i\, \mu^i = 1$, and g_{mn} is the metric on the 2-sphere scaled to radius $1/(2\sqrt{2}\, g)$. If this ansatz is substituted into the equations of motion following from the Lagrangian

$$\mathcal{L} = \hat{R}\, \hat{*}\mathbb{1} - \tfrac{1}{4}\hat{*}d\hat{\phi} \wedge d\hat{\phi} - \tfrac{1}{2}e^{\hat{\phi}}\, \hat{*}\hat{H}_{(3)} \wedge \hat{H}_{(3)} - \tfrac{1}{2}e^{\frac{1}{2}\hat{\phi}}\, \hat{*}\hat{F}_{(2)} \wedge \hat{F}_{(2)} - 8g^2\, e^{-\frac{1}{2}\hat{\phi}}\, \hat{*}\mathbb{1} \qquad (2.6)$$

of the Salam-Sezgin theory, where $\hat{F}_2 = d\hat{A}_{(1)}$, $\hat{H}_{(3)} = d\hat{B}_{(2)} + \frac{1}{2}\hat{F}_{(2)} \wedge \hat{A}_{(1)}$, one obtains the equations of motion following from (2.1), where $H_{(3)} = e^{2\phi} *d\sigma$ [10].

2.1 The Lorentzian Theory

In order to maintain the supersymmetry in the bosonic sector, the spacetime must admit Killing spinors for which the variations (2.2)-(2.4) of the fermionic fields are zero. Assuming that ϵ is a commuting Majorana spinor, the only nontrivial bilinears of ϵ are $K_\mu = \bar{\epsilon}\gamma_\mu\epsilon$ and $\bar{\epsilon}\gamma_{\mu\nu}\epsilon$ and it turns out that the vector K_μ can be used to completely characterize the solutions. From the vanishing of the right hand side of (2.4) it can be deduced that K_μ must be a covariantly constant null vector:

$$\nabla_\mu K_\nu = 0, \qquad K^\mu K_\mu = 0, \qquad (2.7)$$

and consequently, the metric must be that of a pp-wave (see [11], pg.380-381):

$$ds^2 = 2dudv + H(u,x,y)du^2 + dx^2 + dy^2. \qquad (2.8)$$

The vanishing of the right hand side of (2.2) implies that

$$F^i_{\mu\nu} K^\nu = \star F^i_{\mu\nu} K^\nu = 0, \tag{2.9}$$

where $\star$ is the Hodge dual. Defining the basis one-forms as $e^+ = du$, $e^- = dv + \frac{1}{2} H\, du$, $e^a = dx^a$, where $x^a = (x, y)$ and $ds^2 = 2e^+ e^- + e^a e^a$, the non-vanishing frame components of the torsion-free spin connection and the curvature are given by

$$\omega_{+a} = \tfrac{1}{2}\partial_a H\, e^+ \,, \qquad R_{+a+b} = -\tfrac{1}{2}\partial_a \partial_b H \,, \qquad R_{++} = -\tfrac{1}{2}\nabla^2 H \,. \tag{2.10}$$

The generalization of Coleman's non-Abelian plane waves [1] to curved spacetime now shows that there is a gauge in which the Yang-Mills potential can be chosen as [2]

$$A^i = \mathcal{A}^i(u, x, y)\, du \,. \tag{2.11}$$

In this gauge the Yang-Mills field strength is $F^i = \partial_a \mathcal{A}^i\, dx^a \wedge du$. Notice that although the nonlinear term $A \wedge A$ in the Yang-Mills curvature vanishes the solutions will have a full non-Abelian character as long as the holonomy of the Yang-Mills connection is not contained in any $U(1)$ subgroup of $SU(2)$. The vanishing of $A \wedge A$ just reflects the fact that the non-Abelian pp-waves constitute their own linearized approximation. This is a well-known property of the vacuum pp-waves.

Using the pp-wave curvature given in (2.10) it can be inferred from (2.4) that $\gamma^+ \epsilon = 0$ and $d(e^\phi d\sigma)\gamma_5 \epsilon = 0$. The second condition allows us to write $e^\phi d\sigma = d\lambda$, where λ is an arbitrary function. On the other hand, since (2.3) implies $K^\mu \partial_\mu \phi = K^\mu \partial_\mu \sigma = 0$, one can deduce from (2.3) that the dilaton ϕ and the axion σ can only be arbitrary functions of u. Therefore, $\lambda = \lambda(u)$.

It remains to check whether the field equations are satisfied. Since $\phi = \phi(u)$, $\sigma = \sigma(u)$ and the scalar invariants of the Yang-Mills curvature vanish, the dilaton and the axion field equations are trivially satisfied. The Yang-Mills equations reduce to

$$(\partial_x^2 + \partial_y^2)\mathcal{A}^i = 0 \,. \tag{2.12}$$

Defining $z = x + iy$, we may solve this equation by writing

$$\mathcal{A}^i = \frac{1}{2}[\chi^i(u, z) + \bar{\chi}^i(u, \bar{z})] \,, \tag{2.13}$$

where the $su(2)$ valued functions $\chi^i(u, z)$ are holomorphic in z, and arbitrary in u.

The only non-vanishing frame component of the energy-momentum tensor for this field configuration is $T_{++} = \frac{1}{2}[(\dot{\phi}^2 + \dot{\lambda}^2) + e^{-\phi}\partial_a \mathcal{A}^i\, \partial_a \mathcal{A}^i]$, where dot denotes the differentiation

with respect to u. Using (2.10) the Einstein equations become $-4\partial\bar{\partial}H = (\dot{\phi}^2 + \dot{\lambda}^2) + e^{-\phi}\partial\chi^i\,\bar{\partial}\bar{\chi}^i$, where $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial\bar{z}$. It follows that the metric function $H(u, z, \bar{z})$ is given by

$$H(u, z, \bar{z}) = K(u, z) + \bar{K}(u, \bar{z}) - \tfrac{1}{4}[e^{-\phi}\chi^i(u, z)\bar{\chi}^i(u, \bar{z}) + (\dot{\phi}^2 + \dot{\lambda}^2)z\bar{z}]\,, \qquad (2.14)$$

where $K(u, z)$ is an arbitrary function of u but holomorphic in z.

With ϕ and σ arbitrary functions of u, it can be checked that the supersymmetry constraints from the transformation rules (2.2) - (2.4) are now all satisfied except the condition

$$d\epsilon = \tfrac{\mathrm{i}}{4}\,d\lambda\,\gamma_5\epsilon, \qquad (2.15)$$

which follows from (2.4). This equation can be integrated to fix the final form of the Killing spinor:

$$\epsilon = e^{\frac{\mathrm{i}}{4}\,\lambda(u)\,\gamma_5}\,\eta\,, \qquad (2.16)$$

where $d\eta = 0$. The constant spinor η obeys

$$\gamma^+\eta = 0, \qquad (2.17)$$

showing that the solutions possess 1/2 of the Poincaré supersymmetry.

2.2 The Riemannian Theory

This is most easily obtained by setting $\sigma = i\Sigma$, with Σ real and inserting the appropriate factor of i in the $F^i \wedge F^i$ term in the Lagrangian. The scalar fields ϕ, Σ then take their values in two-dimensional de-Sitter or Anti-de-Sitter space rather than two-dimensional hyperbolic space or the upper-half- plane as it does for the Lorentzian theory. The indefinite metric on the target space is

$$d\phi^2 - e^{2\phi}d\Sigma^2. \qquad (2.18)$$

In the Euclidean-signatured theory there are no Majorana spinors, and so we take the spinors to be chiral. From (2.2), and choosing $\epsilon = +\gamma_5\epsilon$, we see that supersymmetry then requires that F^i must be anti-self-dual

$$F^i = - \star F^i. \qquad (2.19)$$

We easily see from (2.3) that

$$\partial_\mu\phi + e^\phi\partial_\mu\Sigma = 0. \qquad (2.20)$$

The geometrical meaning of (2.20) is that the image of the map $\phi(x), \Sigma(x)$ in the two-dimensional internal space is a lightlike geodesic in de-Sitter or anti-de-Sitter spacetime. From (2.4) we deduce that

$$\epsilon = e^{-\frac{1}{4}\phi}\eta, \tag{2.21}$$

where η is covariantly constant. It follows that the metric must be anti-self-dual or Hyper-Kähler

$$R_{\mu\nu\alpha\beta} = -\frac{1}{2}\epsilon_{\mu\nu\sigma\tau}R^{\sigma\tau}{}_{\alpha\beta}\,. \tag{2.22}$$

In particular the Ricci tensor must vanish. This is consistent with the field equations because the stress tensor of a self-dual Yang-Mills field and of a map with totally light like image vanish identically. It remains to check the equation of motion for the scalar fields. A short calculation reveals that it reduces to the linear equation

$$\nabla^2 e^{\phi} = -\frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu}, \tag{2.23}$$

where ∇^2 is the scalar Laplace operator on the Hyper-Kähler 4-manifold.

A simple example is to take the four-dimensional metric to be flat, which we write as $ds_4^2 = dr^2 + \frac{1}{4}r^2\,\sigma_i^2$, where σ_i are left-invariant one-forms on $SU(2)$, and the Yang-Mills fields to be those of the single-instanton solution. This can be written as

$$A^i = \frac{1}{1+\lambda\,r^2}\,\sigma_i\,, \tag{2.24}$$

where λ sets the scale size of the instanton. It is straightforward to integrate (2.23), yielding

$$e^{\phi} = c + \frac{m}{r^2} + \frac{2\lambda\,(2+\lambda\,r^2)}{(1+\lambda\,r^2)^2}\,, \tag{2.25}$$

where c and m are arbitrary constants of integration. The solution is non-singular at $r=0$ if $m=0$.

3 Non-Abelian AdS pp–waves

One can readily generalize the discussion of section (2.1) to the case when a negative cosmological constant term is present. Consider adding a term $-2\Lambda * \mathbb{1}$ to the action (2.1), where Λ is constant. This will bring in a cosmological constant term in Einstein's equations and the appropriate ansatz for the metric that modifies (2.8) is now

$$ds^2 = \frac{a^2}{z^2}\left(2dudv + H(u,x,z)du^2 + dx^2 + dz^2\right), \tag{3.1}$$

where the constant a is proportional to the radius of curvature. Note that in this section z is used to denote a real coordinate, and should not be confused with the complex coordinate used in section 2.1. The metric (3.1) is related to (2.8) simply by a conformal factor. On each (holographic) slice $z=$ constant, it describes a wave propagating at the speed of light along the horosphere [19, 20].

Assuming that $\phi = \phi(u)$, $\sigma = \sigma(u)$ and using the gauge (2.11) for A^i, the Yang-Mills equations are still given by (2.12) and admit the same solution as in section 2.1. Once again the scalar invariants of the Yang- Mills field vanish and the dilaton and the axion field equations put no restrictions on the arbitrary functions $\phi(u)$ and $\sigma(u)$. The metric (3.1) is a solution of the generalized Einstein equations

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} + \frac{1}{2}\, e^{-\phi} \left(F^i_{\alpha\gamma} F^{i\ \gamma}_{\beta} - \frac{1}{4} F^i_{\gamma\delta} F^{i\ \gamma\delta}\, g_{\alpha\beta} \right) + \frac{1}{2}\, \nabla_\alpha\phi\nabla_\beta\phi + \frac{1}{2}\, e^{2\phi}\nabla_\alpha\sigma\nabla_\beta\sigma, \quad (3.2)$$

provided that $\Lambda = -3/a^2$ and H satisfies a Siklos equation [22] with source:

$$-(\partial_x^2 + \partial_z^2)H + \frac{2}{z}\partial_z H = e^{-\phi}\frac{z^2}{a^2}\,(\partial A)^2 + \dot{\phi}^2 + \dot{\lambda}^2, \quad (3.3)$$

where $(\partial A)^2 = \partial_z A^i \partial_z A^i + \partial_x A^i \partial_x A^i$ and as in section 2.1, $d\lambda = e^{\phi} d\sigma$. The solutions of (3.3) with zero source are known to be singular at $z = +\infty$ [21].

In view of an AdS/CFT interpretation of the non-Abelian pp–waves it may be useful to consider the same construction in all dimensions $D \geq 4$. The metric is

$$ds^2 = \frac{a^2}{z^2}\left(2dudv + H(u, x_m, z)du^2 + h_{mn}dx^m dx^n + dz^2\right), \quad (3.4)$$

where we allow a Ricci flat metric h_{mn} on the $D-3$-dimensional transverse space. We keep the same functional dependence of the scalar fields on the coordinates: $\phi = \phi(u)$ and $\sigma = \sigma(u)$. The definition of the Yang-Mills potential is also unchanged but its equation of motion becomes

$$(\nabla_\perp^2 + \partial_z^2 - (D-4)\frac{1}{z}\partial_z)A^i = 0, \quad (3.5)$$

where $\nabla_\perp^2$ is the Laplace operator of h_{mn}. In D-dimensions the cosmological constant is $\Lambda = -(D-1)/a^2$ and the Siklos equation with source generalizes to

$$-\nabla_\perp^2 H - z^{D-2}\partial_z\left(\frac{1}{z^{D-2}}\partial_z H\right) = e^{-\phi}\frac{z^2}{a^2}\,(\partial A)^2 + \dot{\phi}^2 + \dot{\lambda}^2. \quad (3.6)$$

where $(\partial A)^2 = h^{mn}\partial_m A^j \partial_n A^j + \partial_z A^j \partial_z A^j$ and the gauge group is arbitrary.

4 Non-Abelian pp-waves in General $D = 4$ Supergravities

In this section we show that the non–Abelian pp–waves considered above are supersymmetric solutions of the general class of the four-dimensional supergravity theories describing matter with arbitrary potential coupled to Yang-Mills fields. The model of supergravity interacting with scalar multiplets is given in [14], while the coupling with Yang-Mills fields was first constructed in [15,16]. The bosonic sector of the theory is described by the Lagrangian

$$\begin{aligned}\mathcal{L} &= R *\mathbb{1} - 2\, G^i{}_j *Dz_i \wedge D\overline{z}^j - Ref_{ab} *F^a \wedge F^b - \, Imf_{ab}\, F^a \wedge F^b \\ &- \; g^2\, Ref^{-1}_{ab}\, G^i\, (T_a)_i{}^j\, z_j\, G^k\, (T_b)_k{}^l\, z_l + 2\, e^G \, \left(3 - G_i\, (G^{-1})^i{}_j\, G^j\right) *\mathbb{1}, \end{aligned} \tag{4.1}$$

where $z_i = A_i + i\, B_i$ are complex scalar fields in a realization of the gauge group whose Yang-Mills field strength is $F^a_{\mu\nu}$ and $D_\mu z_i$ is the gauge covariant derivative. Moreover, g denotes the gauge coupling constant, $G(z, \overline{z})$ is the Kähler potential, $G^i = \partial G/\partial z_i$, $G_j = \partial G/\partial \overline{z}^j$, $G^i{}_j = \partial^2 G/\partial z_i \partial \overline{z}^j$ and $f_{ab}(z)$ is a set of holomorphic functions.

The Lagrangian (4.1) ostensibly requires that the scalar potential is non-zero. However, we can always add a constant to the Kähler potential G without affecting G_i or $G^i{}_j$. Taking the constant to minus infinity allows us to discard the scalar potential term. In this way, we can view the dimensionally-reduced Salam-Sezgin Lagrangian (2.1) as a special case of (4.1). Note that in this case the gauge covariant derivative on the scalar fields becomes the ordinary derivative, because the $SU(2)$ gauge group does not act on the scalar fields (ϕ, σ). The non-Abelian pp-waves of section 2.1 are therefore solutions in this limit and it is clear that the same procedure can be employed for other gauge groups as well.

In general the fermionic supersymmetry transformations are

$$\delta\lambda^a = -\frac{1}{2}\, F^a_{\alpha\beta}\, \gamma^{\alpha\beta} \epsilon_L + \frac{i}{2}\, g\, Ref^{-1}_{ab}\, G^i\, (T_b)_i{}^j\, z_j \epsilon_L, \tag{4.2}$$

$$\delta\chi_i = -\,\gamma^\alpha D_\alpha\, (A_i + i\, \gamma_5\, B_i)\; \epsilon - \sqrt{2}\, e^{G/2}\, (G^{-1})_i{}^j\, G_j \epsilon\,, \tag{4.3}$$

$$\delta\psi_\alpha = 2\, \nabla_\alpha\, \epsilon + \left(G^i\, D_\alpha z_i - G_i\, D_\alpha \overline{z}^i\right) \epsilon + \, e^{G/2}\, \gamma_\alpha\, \epsilon\,. \tag{4.4}$$

As in section 2.1, it is easy to see that (4.2) implies

$$\iota_V F_a = Re\, X_a\, V, \tag{4.5}$$

$$\iota_V * F_a = Im\, X_a\, V, \tag{4.6}$$

where $X_a = \frac{i}{2}\, g\, Ref^{-1}_{ab}\, G^i\, (T_b)_i{}^j\, z_j$ and V is the null vector associated to the Weyl spinor ϵ_L. In order to have a non-abelian pp-wave solution one has to impose then $X_a = 0$, which implies $G^i = 0$. This cannot be identically true otherwise there would be no kinetic term

for the scalars. Rather, one can ask the scalars to be at an extremal point of the Kähler function

$$\begin{aligned} G^i|_{z=z_0} &= 0, & (4.7) \\ D_\alpha z_i &= 0. & (4.8) \end{aligned}$$

$G^i = 0$ implies that the scalar potential reduces to the constant term $6\,e^G$, which is consistent with the equations of motion for the scalar fields obtained from (4.1) when these fields are covariantly constant. Moreover, (4.3) is then automatically satisfied. The gravitino supersymmetry transformation then implies that the spinorial parameter satisfies $D_\alpha\epsilon + 1/2\,e^{G/2}\gamma_\alpha\,\epsilon = 0$.

The term $6\,e^G$ in the potential acts as a negative cosmological term and the metric is that given in section 3. (Since the scalar fields can at most be certain functions of u and will not explicitly appear in our discussion, we shall continue to use z as the real coordinate of section 3). Defining the frame $e^+ = \frac{a}{z}\,du$, $e^- = \frac{a}{z}\left(dv + \frac{1}{2}H\,du\right)$, $e^3 = \frac{a}{z}\,dz$, $e^4 = \frac{a}{z}\,dx$ so that $ds^2 = 2\,e^+e^- + e^3e^3 + e^4e^4$, we find the non vanishing components of the spin connection to be

$$\begin{aligned} \omega_{+3} &= \tfrac{1}{2}\partial_z H\,du - \tfrac{1}{a}e^-, \\ \omega_{+4} &= \tfrac{1}{2}\partial_x H\,du, \\ \omega_{-3} &= -\tfrac{1}{z}du, \\ \omega_{34} &= \tfrac{1}{z}dx, \end{aligned} \qquad (4.9)$$

and the Ricci tensor can be written as

$$R_{\mu\nu} = -\frac{3}{a^2}\,g_{\mu\nu} + \delta^u_\mu\,\delta^u_\nu\left[-\frac{1}{2}\left(\partial_x^2 H + \partial_z^2 H\right) + \frac{1}{z}\partial_z H\right]. \qquad (4.10)$$

For such solutions the cosmological constant obeys $e^G = \frac{1}{a^2}$ and choosing a to be the positive root, we get $a = e^{-G/2}$. Assuming that Eqs.(4.7), (4.8) hold, the first two supersymmetry variations (4.2), (4.3) vanish provided ϵ satisfies $\gamma^+\epsilon = 0$ as in section 2.1. Setting the variation (4.4) of the gravitino to zero then gives

$$\begin{aligned} \partial_v\epsilon &= 0, \\ \partial_u\epsilon - \tfrac{1}{2z}\,\gamma^{-3}\epsilon &= -\tfrac{1}{2z}\gamma^-\epsilon, \\ \partial_z\epsilon &= -\tfrac{1}{2z}\gamma^3\epsilon, \\ \partial_x\epsilon + \tfrac{1}{2z}\,\gamma^{34}\epsilon &= -\tfrac{1}{2z}\gamma^4\epsilon. \end{aligned} \qquad (4.11)$$

Since $\left\{\gamma^+\,,\,\gamma^3\right\} = 0$ we can project these equations on the two spinors ϵ_1, ϵ_2 such that

$\gamma^3\epsilon_1 = +\epsilon_1$, $\gamma^3\epsilon_2 = -\epsilon_2$. This gives us the two systems

$$\begin{array}{rclrcl}\partial_v\epsilon_1 &=& 0, & \partial_v\epsilon_2 &=& 0,\\ \partial_u\epsilon_1 &=& 0, & \partial_u\epsilon_2 &=& -\frac{1}{z}\gamma^-\epsilon_2,\\ \partial_z\epsilon_1 &=& -\frac{1}{2z}\epsilon_1, & \partial_z\epsilon_2 &=& +\frac{1}{2z}\epsilon_2,\\ \partial_x\epsilon_1 &=& 0, & \partial_x\epsilon_2 &=& -\frac{1}{z}\,\gamma^4\epsilon_2.\end{array} \tag{4.12}$$

The first system of equations has the solution $\epsilon_1 = \frac{1}{\sqrt{z}}\eta$, where η is a constant spinor satisiying $\gamma^+\eta = 0 = (1-\gamma^3)\eta$. The second system of equations instead does not admit a solution. Therefore, the presence of a cosmological constant term implies that the solutions preserve 1/4 of the supersymmetry.

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References

[1] S. Coleman, *Nonabelian plane waves*, Phys. Lett **B70** (1977) 59.

[2] R. Güven, *Solution for gravity coupled to non-Abelian plane waves*, Phys. Rev. **D19** (1979) 474-472.

[3] A. Trautman, *A class of null solutions to Yang-Mills equations*, J. Phys. **A13** (1980) L1.

[4] R. Güven, *Plane waves in effective field theories of superstrings*, Phys. Lett. **B191** (1987) 275-281.

[5] G. W. Gibbons, *Quantized fiels propagating on plane wave space-times*, Commun. Math. Phys. **45** (1975) 191.

[6] S. Deser, *Plane waves do not polarize the vacuum*, J. Phys. **A8** (1975) 1972.

[7] C. Aragone and G. Coliacomo, *Gravity-matter instantons*, Lettere al Nuovo Cimento **31** (1981) 135-139.

[8] G. Etesi and T. Hausel, *On Yang–Mills instantons over multi-centered gravitational instantons*, Commun. Math. Phys. **235** (2003) 275-288, hep-th/0207196.

[9] G. Etesi, *Classification of 't Hooft instantons over multi-centered gravitational instantons*, Nucl. Phys. **B662** (2003) 511-530, hep-th/0303146.

[10] G.W. Gibbons and C.N. Pope, *Consistent S^2 Pauli reduction of six-dimensional chiral gauged Einstein-Maxwell supergravity*, hep-th/0307052.

[11] E. Herlt, C. Hoenselaers, D. Kramer, M. MacCallum, H. Stephani in: Exact Solutions of Einstein's Field Equations, Cambridge University Press, Cambridge (2003).

[12] H. Nishino and E. Sezgin, *Matter and gauge couplings of $N = 2$ supergravity in six-dimensions*, Phys. Lett. **B144**, (1984) 187.

[13] A. Salam and E. Sezgin, *Chiral compactification on (Minkowski)$\times S^2$ of $N = 2$ Einstein-Maxwell supergravity in six-dimensions*, Phys. Lett. **B147**, (1984) 47.

[14] E. Cremmer, B. Julia, J. Scherk, S. Ferrara, L. Girardello and P. Van Nieuwenhuizen, *Spontaneous symmetry breaking and Higgs effect in supergravity without cosmological constant*, Nucl. Phys. **B147** (1979) 105.

[15] E. Cremmer, S. Ferrara, L. Girardello and A Van Proeyen, *Coupling supersymmetric Yang-Mills theories to supergravity*, Phys. Lett. **B116** (1982) 231.

[16] E. Cremmer, S. Ferrara, L. Girardello and A Van Proeyen, *Yang-Mills theories with local supersymmetry: Lagrangian, transformation laws and superHiggs effect*, Nucl. Phys. **B212** (1983) 413.

[17] R. Güven, E. Yörük, *Stringy Robinson-Trautman solutions*, Phys. Rev. **D54** (1996) 6413-6423, hep-th/9609078.

[18] G.W. Gibbons, R. Güven and C.N. Pope, *3-branes and uniqueness of the Salam-Sezgin vacuum*, hep-th/0307238.

[19] S. T. C. Siklos, in: Galaxies, axisymmetric systems and relativity, ed. M. A. H. MacCallum, Cambridge University Press, Cambridge (1985).

[20] G. W. Gibbons and P. J. Ruback, *Classical gravitons and their stability in higher dimensions*, Phys. Lett. **B171** (1986) 390-395.

[21] A. Chamblin and G. W. Gibbons, *Supergravity on the brane*, Phys.Rev.Lett. **84** (2000) 1090-1093, hep-th/9909130.

[22] J. Podolsky, *Interpretation of the Siklos solutions as exact gravitational waves in the anti-de Sitter universe*, Class. Quant. Grav. **15** (1998) 719-733, gr-qc/9801052.